Upper bounds for the sizes of finitely generated algebras

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Garrett Birkhoff's 1935 paper "On the structure of abstract algebras" includes as a corollary of his work on free algebras the following upper bound on the cardinality of finitely generated algebras:

Theorem: [Birkhoff] Let $\mathcal{K}$ any set of algebras $\mathbf{A}_1, \ldots, \mathbf{A}_r$. The cardinality of any algebra generated by n elements and obeying the equations of $\mathcal{K}$ is at most

$$\prod_{i=1}^{r} |A_i|^{|A_i|^n}.$$

An appealing aspect of Birkhoff's result is that it is completely general and depends only on the cardinalities of the universes A_i. Subsequently, Sioson gave characterizations of those $\mathcal{K}$ consisting of finite algebras for which Birkhoff's upper bound is obtained for all finite n. He provided characterizations in terms of algebraic properties of the $\mathbf{A}_i$, as well as by means of the term operations of the $\mathbf{A}_i$, and he gave an effective method involving numerical invariants of the $\mathbf{A}_i$ to test if the algebras in $\mathcal{K}$ achieve the upper bound.

In this talk I present general upper bound results in the spirit of Birkhoff's theorem and Sioson's characterization theorems. These upper bounds are widely applicable, are simply stated, and depend on very general and transparent parameters such as the cardinalities of the automorphism groups of the algebras in $\mathcal{K}$, the cardinalities of subalgebras of the $\mathbf{A}_i$, and for a given pair of algebras, how many subalgebras of one algebra are isomorphic to the other. And when the congruence relations of the algebras in $\mathcal{K}$ are well-behaved, the upper bounds involve the sizes of the congruence classes of the congruence relations of the algebras in $\mathcal{K}$. Characterization theorems in the same spirit as those of Sioson are presented for those $\mathcal{K}$ for which the upper bound is obtained. Numerous illustrative examples will be given.